III Knots in Contact Manifolds

recall a knot K (ie embedding of S^1) in a contact manifold (M,ξ) is <u>Legendrian</u> if

$$T_xK \subset \xi_x \quad \text{for all } x \in K$$

let $\nu(K)$ = normal bundle of K
(identify with tubular neighborhood)

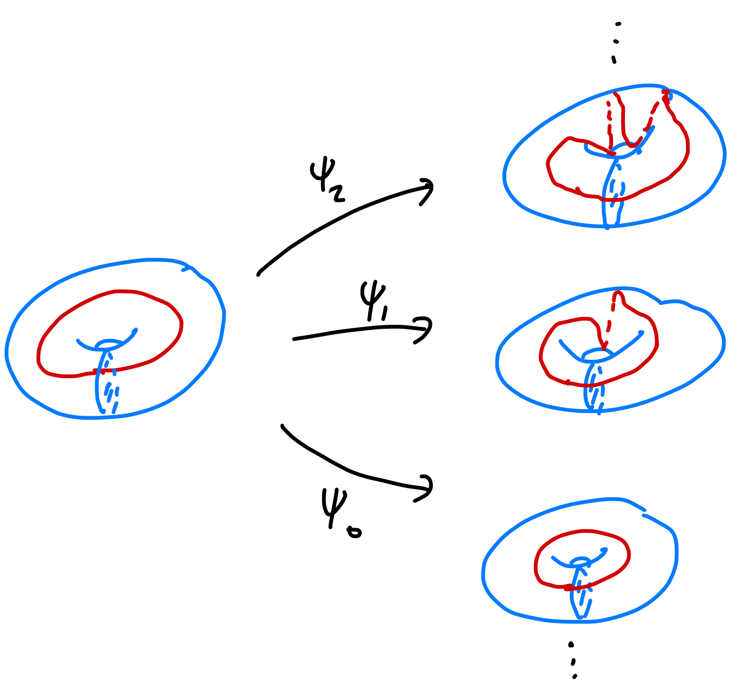
$\nu(K)$ is an $\mathbb{R}^2$-bundle

$$\begin{array}{c} \mathbb{R}^2 \to \nu(K) \\ \quad\quad \downarrow \\ \quad\quad K \end{array}$$

exercise: since M is oriented so is $\nu(K)$ and therefore trivial.

so $\nu(K) \cong S^1 \times \mathbb{R}^2$

upto isotopy there are $\mathbb{Z}$-worth of ways to identify $\nu(K)$ with $S^1\times\mathbb{R}^2$ that differ by "twisting



$\psi_n : S^1\times D^2 \to S^1 \times D^2$
$(\phi,(r,\theta)) \mapsto (\phi,(r,\theta+n\phi))$

so if $f: S^1 \times D^2 \to \nu(K)$ one trivialization
then $\Psi_n^{-1} \circ f$ gives $\mathbb{Z}$-worth

exercise: Show these are only trivializations
up to isotopy

an identification of $\nu(K)$ with $S^1 \times \mathbb{R}^2$ is called a
framing of K

a non zero section s of $\nu(K)$ gives a framing of K

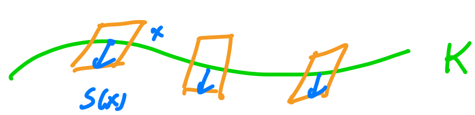
can see this by picking another section $\tilde{s}$ of $\nu(K)$ that
is transverse to s

so $s(x), \tilde{s}(x)$ is a basis for $\nu_x(K)$

then
$$\Psi : K \times \mathbb{R}^2 \to \nu(K)$$
$$(x,(a,b)) \mapsto a\, s(x) + b\, \tilde{s}(x)$$
is a trivialization

If K is a Legendrian knot we get a framing:

let $x \in K$, set $s(x) \in \xi_x \cap \nu(K)$
s.t. $s(x) \neq 0$



this is called the contact framing of K and denoted
$\mathcal{F}(K,\xi)$

exercise: if X_α is a Reeb vector field for ξ then
this also frames K, show this gives

same framing as above

if K is null-homologous then there is an embedded surface $\Sigma \subset M$ such that $\partial \Sigma = K$

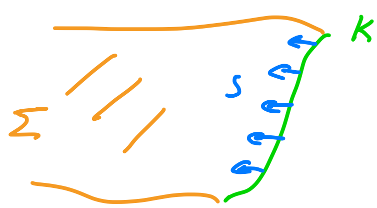
exercise: prove this

Σ is called a Seifert surface for K

(easy to construct Σ in $\mathbb{R}^3$, S^3)

given Σ we get a framing for K:

$x \in K$, let $s(x) \neq 0$ in $T_x \Sigma \cap \nu_x K$



this is called the Seifert framing of K

exercise: this framing is well-defined

given two framings $\mathcal{F}_1$ and $\mathcal{F}_2$ of K we can associate an integer

$$S^1 \times \mathbb{R}^2 \xrightarrow{\mathcal{F}_1} \nu(K) \xleftarrow{\mathcal{F}_2} S^1 \times \mathbb{R}^2$$

so $\mathcal{F}_2^{-1} \circ \mathcal{F}_1 : S^1 \times \mathbb{R}^2 \to S^1 \times \mathbb{R}^2$

is isotopic to ψ_n some n (ψ_n defined above)

so we say "$\mathcal{F}_1 - \mathcal{F}_2 = n$"

the <u>Thurston-Bennequin invariant of K</u> is

$$\boxed{tb(K) = \mathcal{F}(K,\xi) - \text{Seifert framing}}$$

contact framing

<u>exercise</u>: if K_1 isotopic to K_2 through Legendrian knots, then $tb(K_1) = tb(K_2)$

now suppose K is an oriented Legendrian knot and $K = \partial\Sigma$

<u>exercise</u>: prove an oriented 2-plane bundle over a surface with boundary is trivial

so $\xi|_\Sigma = \Sigma \times \mathbb{R}^2$

and $\xi|_K$ inherits a trivialization

$$\xi|_K \cong K \times \mathbb{R}^2$$

comming from $\Sigma \times \mathbb{R}^2$

<u>exercise</u>: the trivialization $\xi|_\Sigma$ is <u>not</u> unique but when restricted to $\partial\Sigma$ it is unique

since K is oriented we can choose a vector $v(x)$ at $x \in K$ that points in direction of orientation

so v gives a section

$$\begin{tikzcd} K \times \mathbb{R}^2 \arrow[d] \\ K \arrow[u, bend right, "v"'] \end{tikzcd}$$

think of $v: K \to \mathbb{R}^2$

this map is non-zero so it has a

winding number about origin

let $\boxed{r(K) = \text{winding number of } r \text{ about origin}}$

this is called the rotation number of K

exercise: if K_1 isotopic to K_2 through oriented Legendrian knots,

then $r(K_1) = r(K_2)$

the "classical invariants" of an oriented Legendrian knot are

1) knot type
2) Thurston-Bennequin invariant
3) rotation number

let's see how to compute these invariants in $(\mathbb{R}^3, \xi_{std})$

recall $\xi_{std} = \ker(dz - y\,dx)$

the front projection is

$$\pi: \mathbb{R}^3 \to \mathbb{R}^2 : (x,y,z) \mapsto (x,z)$$

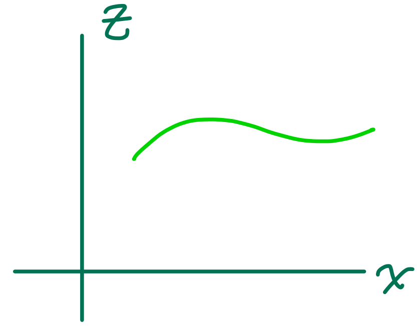
if K is a Legendrian knot in $\mathbb{R}^3$ we can parameterize it

$$\psi: S^1 \to \mathbb{R}^3 : \theta \mapsto (x(\theta), y(\theta), z(\theta))$$

$$K \text{ Legendrian} \Leftrightarrow \underset{\underset{z'd\theta - y x' d\theta}{\shortparallel}}{\psi^*\alpha} = 0 \Leftrightarrow z' - y x' = 0$$

consider $\pi \circ \Psi : S^1 \to \mathbb{R}^2$

where z-coordinate is a function of x-coordinate



$$\frac{\frac{dz}{d\theta}}{\frac{dx}{d\theta}} = \frac{dz}{dx} \quad \text{so}$$

for Legendrian K $\boxed{y(\theta) = \frac{dz}{dx}(\theta)}$

you can recover the y-coordinate from the slope in the xz-plane

note: if $x'(\theta_0) = 0$, then $z'(\theta_0) = y(\theta_0)\, x'(\theta_0) = 0$

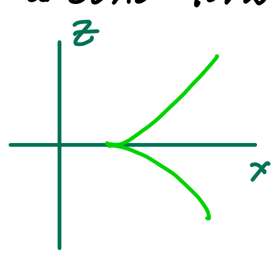
so z' always vanishes to at least the order of x'

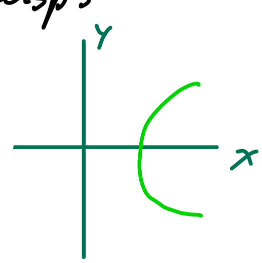
so $\lim_{\theta \to \theta_0} \frac{z'(\theta)}{x'(\theta)}$ will exist

(i.e. always have a y-coordinate)

note: this implies <u>no vertical tangents</u>

get around this with cusps





generically: $\theta \mapsto (\theta, \frac{3}{2}\theta^{1/2}, \theta^{3/2})$

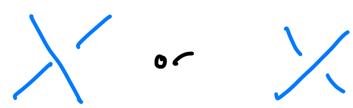
"semi-cubical cusp"

K is still smooth even though $\pi(K)$ not so an immersed curve with no vertical tangencies in $\mathbb{R}^2$ ← xz-plane

determines a Legendrian knot by setting $y = \frac{dz}{dx}$

exercise: picture these knots
(e.g. draw xy-projection)

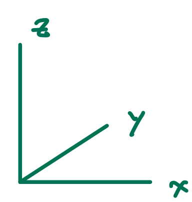
note: knot diagrams usually have crossing information

e.g. 

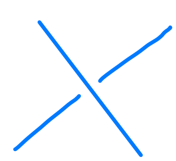
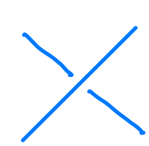
but this comes for free here since $y = \frac{dz}{dx}$

so y-coordinate is bigger if slope bigger

recall for a right handed coordinate system

we need 

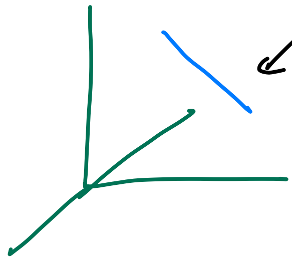
so for a Legendrian knot always see

 never 

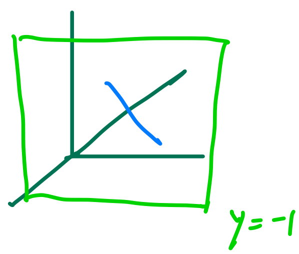
lemma 1: $\boxed{\text{any arc } A \text{ in } (M,\xi) \text{ can be } C^0\text{-isotoped, rel end points to a Legendrian arc (and rel any points where } A \text{ already Legendrian)}}$

Proof: start in $(\mathbb{R}^3, \xi_{std})$

eg.

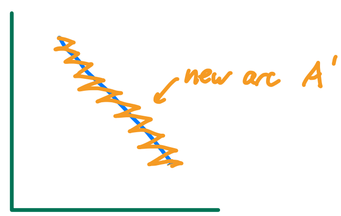


if we took "Legendrian Lift" (set $y = \frac{dz}{dx}$) we would get a Legendrian but with $y = -1$



so <u>not</u> C^0-close one rel end points

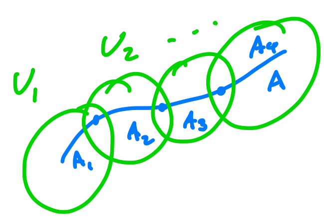
but replace $\pi(A)$ with arc with zig-zags so all slopes in $(-\varepsilon, \varepsilon)$



Legendrian lift of A' C^0-close to A!

exercise: prove for any $A \subset (\mathbb{R}^3, \xi_{std})$

now given $A \subset (M, \xi)$ cover A by Darboux balls



break A into pieces $A_1 \dots A_n$

so that each $A_j \subset$ Darboux ball

approximate A_1 in $U_1 \cong (\mathbb{R}^3, \xi_{std})$ by above

then A_2 (rel A_1) ...

(to deal with smoothness make A_j overlap)

Corollary 2:

$$\boxed{\text{every knot in } (M,\xi) \text{ can be } C^0\text{-approximated by a Legendrian knot}}$$

lets compute $tb(K)$ and $r(K)$ in front projection

Fact:

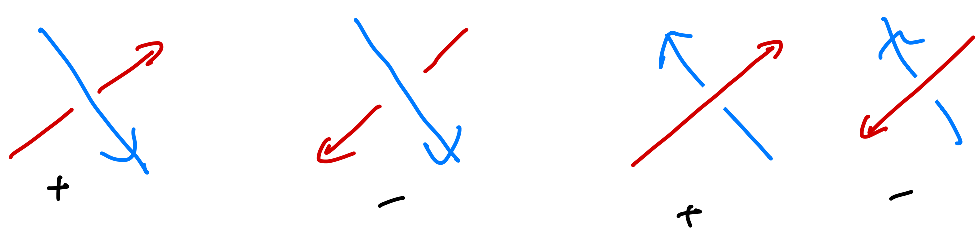
if $\mathcal{F}$ a framing on a knot K and $K = \partial\Sigma$

linking number

then $\mathcal{F}$ - Seifert framing $= \text{link}(K, K')$

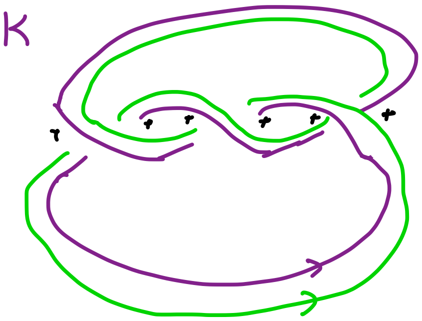
where K' is a copy of K pushed in the direction of a vector field along K defining $\mathcal{F}$

for oriented links K, K' one assigns a sign ε to each crossing of K and K'



$$\text{link}(K,K') = \frac{1}{2} \sum_{\substack{\text{all crossing} \\ \text{between } K, K'}} \varepsilon(c)$$

example: K

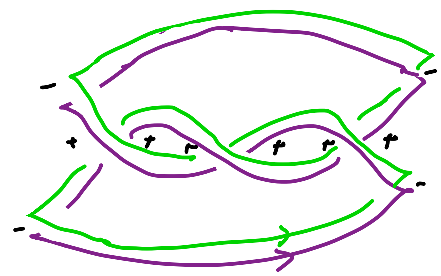


so $\text{link}(K, K') = \frac{1}{2}(6) = 3$

so for $tb(K)$ if K Legendrian let $K' = K$ pushed in Reeb direction

so $tb(K) = \text{link}(K, K')$

example:



$tb(K) = \frac{1}{2}(6-4) = 1$

given a knot diagram D for K we say the writhe of D is: orient D

$$\text{writhe}(D) = \sum_{\text{crossings } c} \varepsilon(c)$$

exercise: if K is Legendrian in $(\mathbb{R}^3, \xi_{std})$ then show

$$\boxed{tb(K) = \text{writhe}(\pi(K)) - \#\text{ left cusps}}$$

hint: consider last example

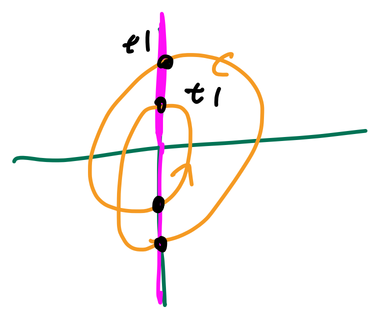
now for rotation number

we can trivialize $\xi_{std} = \ker(dz - y\,dx)$
$= \mathrm{span}\{ \frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y\frac{\partial}{\partial z} \}$

by $\frac{\partial}{\partial y}, \frac{\partial}{\partial x} + z\frac{\partial}{\partial y}$

to count a winding number in $\mathbb{R}^2$ one fixes a line and counts how many times (+ counterclockwise, − clockwise) you pass this direction, then divide by 2

e.g.

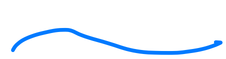


winding +2

note we do not need a Seifert surface to compute $r(K)$ in $\mathbb{R}^3$ since can globally trivialize ξ.

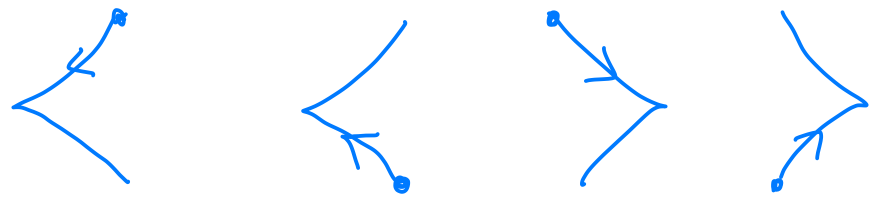
to compute $r(K)$, choose $\frac{\partial}{\partial y}$ in ξ_{std} and count how many times oriented tangent vector crosses this

at points like

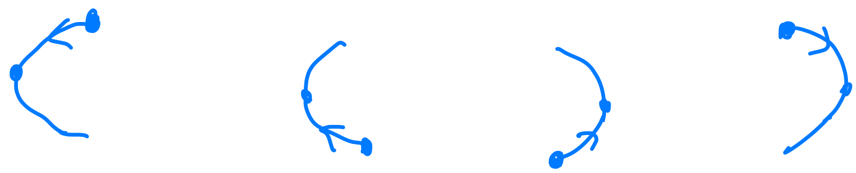


tangent vecto has x-component so does not pass y-direction can assume all cusps are horizontal ($\prec$ not $\curlyvee$) why?

at a cusp we see



↓ in xy-plane

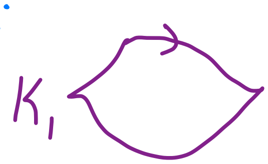


so tangen vecto

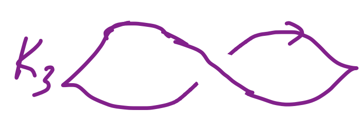
so we see

$$\boxed{r(K) = \frac{1}{2}(\#\text{ down cusps} - \#\text{ up cusps})}$$

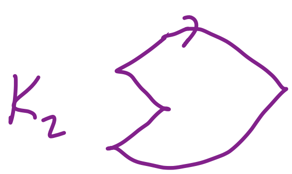
examples:



$tb = -1$
$r = 0$



$tb = -2$
$r = 1$



$tb = -2$
$r = 1$

so K_1 <u>not</u> Legendrian isotopic to K_2 or K_3

are K_2 and K_3 isotopic?

<u>Major Line of Research</u>

fix (M,ξ) and smooth oriented knot $K \subset M$

let $\mathcal{L}(K) = \{$ Legendrian isotopy classes of Legendrian knots in (M,ξ) smoothly isotopic to $K \}$

consider map $\Phi : \mathcal{L}(K) \to \mathbb{Z}^2$

$L \longmapsto (r(L), tb(L))$

Determine • image Φ called the <u>geography problem</u>

• for each $(r,t) \in$ image Φ what is $\Phi^{-1}(r,t)$

called the <u>botany problem</u>

<u>note:</u> • Cor 2 says $\mathcal{L}(K) \neq \emptyset$

• given

called <u>positive/</u> <u>negative</u> <u>stabilization</u>

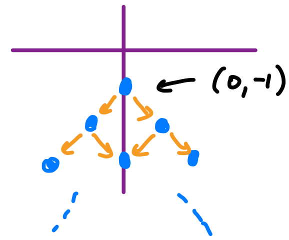
clearly $tb(S_\pm(K)) = tb(K) - 1$

$r(S_\pm(K)) = r(K) \pm 1$

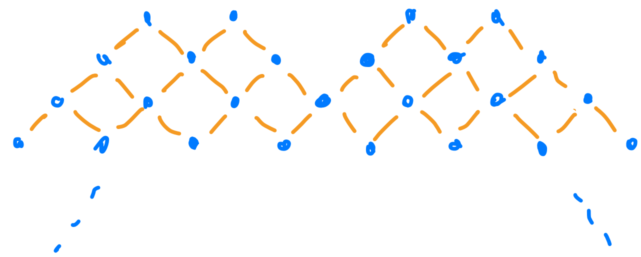
so $\mathcal{L}(K)$ has infinitely many elts!

example: for the unknot U in $(\mathbb{R}^3, \xi_{std})$ we will see

$\text{Im}\,\Phi$ is



for other knots might see



so $\text{im}\,\Phi$ called the mountain range of K

We now turn to transverse knots.

recall a knot K in (M, ξ) is transverse if $T_xK \pitchfork \xi_x$ for all $x \in K$

note: we are taking ξ to be oriented

if K is positively transverse to ξ we say it is a positive transeverse knot

similarly for negative transverse knots

we will only consider positive transvers knots so that is what transverse will mean

assume K is null-homologous, so $K = \partial\Sigma$ some surface Σ in M

as above $\xi|_\Sigma = \Sigma \times \mathbb{R}^2$

let v be a non-zero section of $\xi|_\Sigma$

push K along v to get a copie K' of K

define the self-linking of K to be

$$\boxed{sl(K) = \text{link}(K, K')}$$

note: we only defined link in $\mathbb{R}^3$, in general

$$\text{link}(K, K') = L \cdot \Sigma$$

algebraic intersection number

consider $\mathbb{R}^3 / x \mapsto x+1$ with contact structure $\xi_{std} = \ker(dz - y\,dx)$

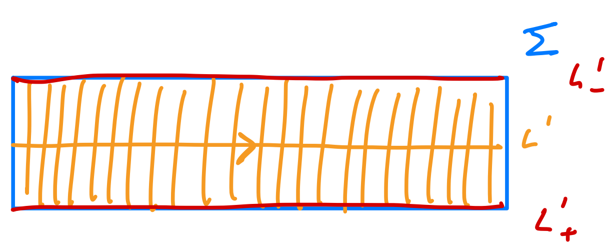
$L' = \{(x,0,0)\} \in \mathbb{R}^3 / x \mapsto x+1$

is a Legendrian knot

given any Legendrian knot L in (M, ξ)

Th^{m} II.4 says L has a neighborhood N that is contactomorphic to a neigh. N' of L'

note: in N' we see $\Sigma = \{(x,y,0) \mid |y| < \varepsilon\}$



L'_+ is a transverse knot

its image L_+ in N is called the transverse push off of L

lemma 3:

$\boxed{\text{any knot in } (M,\xi) \text{ can be } C^0\text{-isotoped to be a transverse knot}}$

Proof: Corollary 2 + transverse push off ∎

exercise: for a Legendrian knot K

Show
$$sl(K_+) = tb(K) - r(K)$$